How quickly calculate VaR for the whole portfolio?

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The portfolio P is composed of N instruments, the value of the portfolio at time t is:

$$P_t = \sum_{i=1}^{N} p_i * x_t \qquad (1)$$

where x_t is the price of the instrument, t is the time, and p_i is the amount of shares of a instrument i in the portfolio.

Risk of the portfolio by Markovitz'a is:

$$Risk_p = \sigma_p^2 = \sum_{i=1}^{N} p_i^2 \sigma_i^2 \qquad (2)$$

where σ_i is the standard deviation of historical returns of price of the instrument.
The above equation can be approximated replaced as follows:

$$Risk_p = \sigma_p^2 = \sum_{i=1}^{N} p_i^2 \int_{-\infty}^{\infty} f_{y_i} y_i^2 \, dy_i \qquad (3)$$

where f_{y_i} is the probability distribution function of returns. The Value at Risk, abbreviated VaR, is as follows:

$$VaR_p = \sum_{i=1}^{N} p_i^2 \int_{-\infty}^{-\beta_i} f_{y_i} y_i^2 \, dy_i \qquad (4)$$

Value $\frac{\beta_i}{\sigma_i}$ is the same for all instruments, if the prices of these instruments returns are independent of each other. Otherwise, the threshold β_i should be calculated from the covariance matrix.

Suppose t-Student distribution with degrees of freedom α_i for price returns of the instrument i.
t-Student distribution is as follows:

$$P_\alpha(y) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\,\Gamma(\alpha/2)\left(1+\frac{y^2}{\alpha}\right)^{\frac{\alpha+1}{2}}} \qquad (5)$$

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \qquad (6)$$

The degrees of freedom α can be computed by Maximum Liklihood Method.

In the following discussion we will use the Objective Value (Ref [1]). For the t-Student distribution it is as follows:

$$w(y)=\frac{\delta+1}{2}\ln(1+\frac{y^2}{\delta}) \quad (7)$$

We calculate the average value <w> for each instrument. The Objective Value for the Normal distribution is the variance, so we can use the table for a normal distribution to calculate δ. The example of table can be found here http://www.statystyka-zadania.pl/tablica-rozkladu-normalnego/ . We do this so that VaR 5%, we have β = 1.65 σ. So the relationship between <w> and β are:

$$\beta=1.65\sqrt{\langle w\rangle} \quad (8)$$

We calculate the average value <w> as follows:

$$\langle w_i\rangle=\sum_{j=0}^{t}\frac{\alpha_i+1}{2}\ln(1+\frac{y_{i,j}^2}{\alpha_i}) \quad (9)$$

where i is the index of the instrument, and j is the time.

We still have to take into account the correlations between the returns of financial instruments. The publication [1] described how to calculate covariance matrix taking into account the objective value.
We substitute into the formula (8) the average value of w: <w>, and compute the degrees of freedom **β** matrix. We calculate the eigenvalues and eigenvectors for the matrix **β** using **Singular Value Decomposition**. We already have vectors that are independent in space and we can use equation (4).
The final formula for VaR for the portfolio include:

$$VaR_p=\sum_{i=1}^{K}\int_{-\infty}^{-1.65*v_i} f_{y_i} y_i^2\, dy_i \quad (11)$$

where v_i is the eigenvalue of the covariance matrix, K the number of significant eigenvalues of the covariance matrix, and f_{y_i} we take from eq. (5).

Bibliography:

[1] K. Urbanowicz, P. Richmond and J.A. Hołyst, Risk evaluation with enhanced covariance matrix, Physica A , doi:10.1016/j.physa.2007.05.034, (2007).